15.9: How to decide what change of variables to use?

Unfortunately, no hard and fast rule for this, but sometimes there is a "natural choice":

e.g. b/c it makes the region much simpler

and/or it makes the integrand much simpler.

Example 1:

Stewart 15.9.25:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$$

R: trapezoid bdd by $(1,0), (2,0), (0,2), (0,1)$.

Solution: Natural choice for u and v is

$$u = y - x \qquad v = y + x$$

Solve: $x = \dfrac{v-u}{2} \qquad y = \dfrac{v+u}{2}$

$\boxed{\text{What happens to the bounds?}}$

(i.e. region of integration)

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$\boxed{\text{What happens to the integrand?}}$

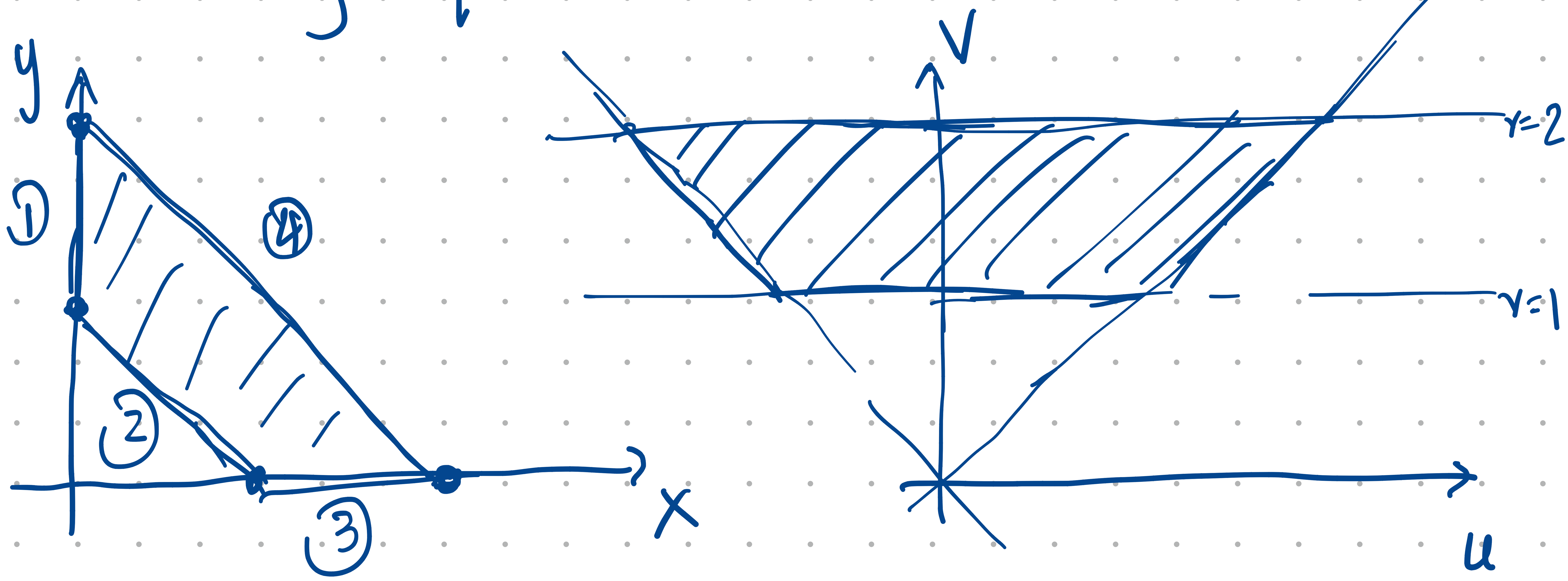
$$\cos\left(\frac{y-x}{y+x}\right) dx\,dy$$

$$\cos\left(\frac{u}{v}\right) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\, du\,dv$$

$$\left|\det\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right| = \frac{1}{2}$$

$$\cos\left(\frac{u}{v}\right)\cdot\left(\frac{1}{2}\right) du\,dv$$

The most systematic way of switching bounds is to describe the region completely algebraically (i.e. using inequalities) and then substitute.



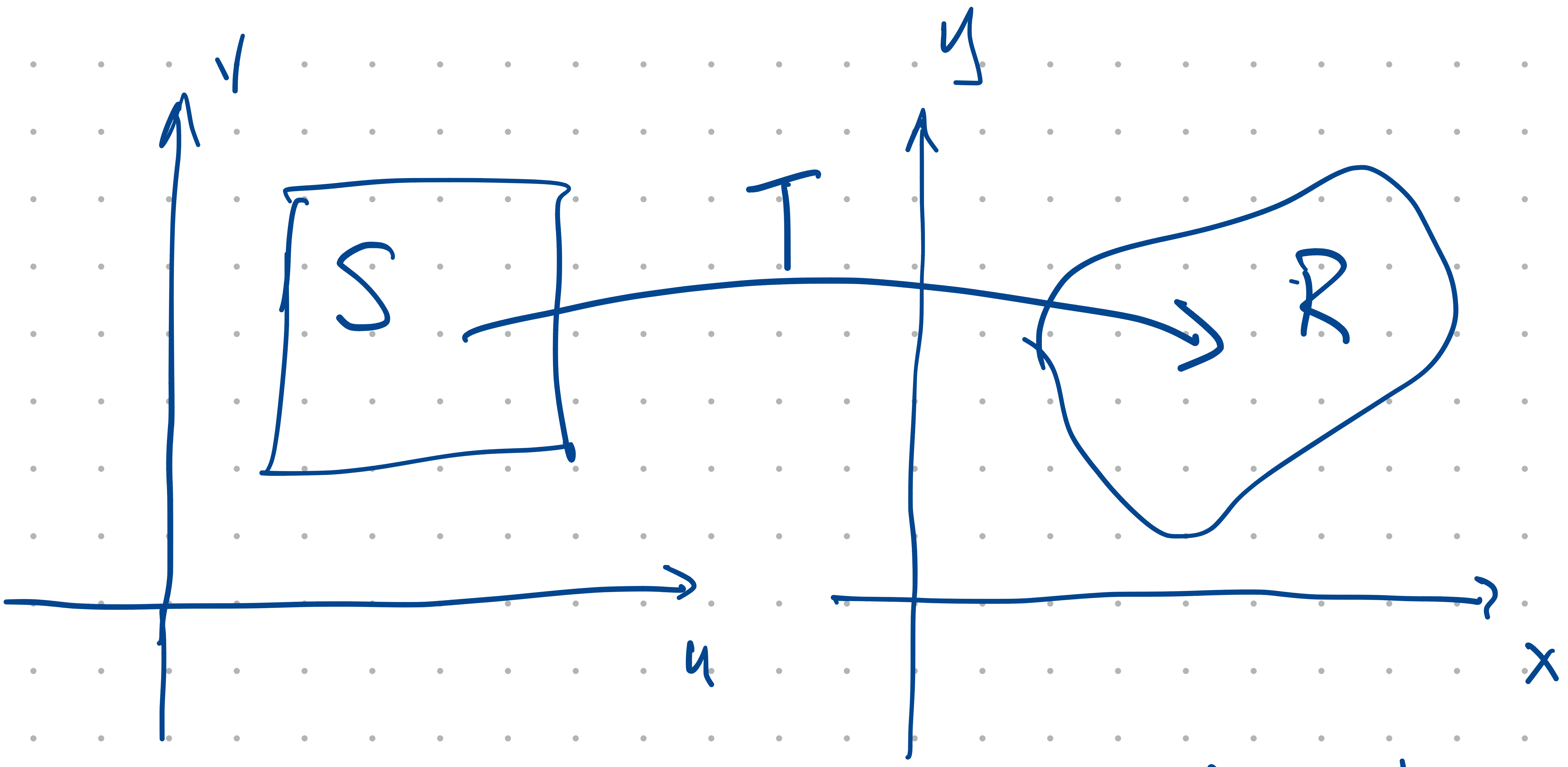
① $x \geq 0$ $\qquad\qquad$ $\frac{v-u}{2} \geq 0$

② $x+y \geq 1$ $\quad\rightsquigarrow\quad$ $v \geq 1$

③ $y \geq 0$ $\qquad\qquad$ $\frac{v+u}{2} \geq 0$

④ $x+y \leq 2$ $\qquad\qquad$ $v \leq 2$

$$\int_1^2 \int_{-v}^{v} \frac{1}{2} \cos\left(\frac{u}{v}\right) du\, dv$$



T gives x and y in terms of u and v

$$\iint_S \cdots \, du\,dv \longleftarrow \iint_R \cdots \, dx\,dy$$

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy$$

$x = u+v$

$y = u-v$

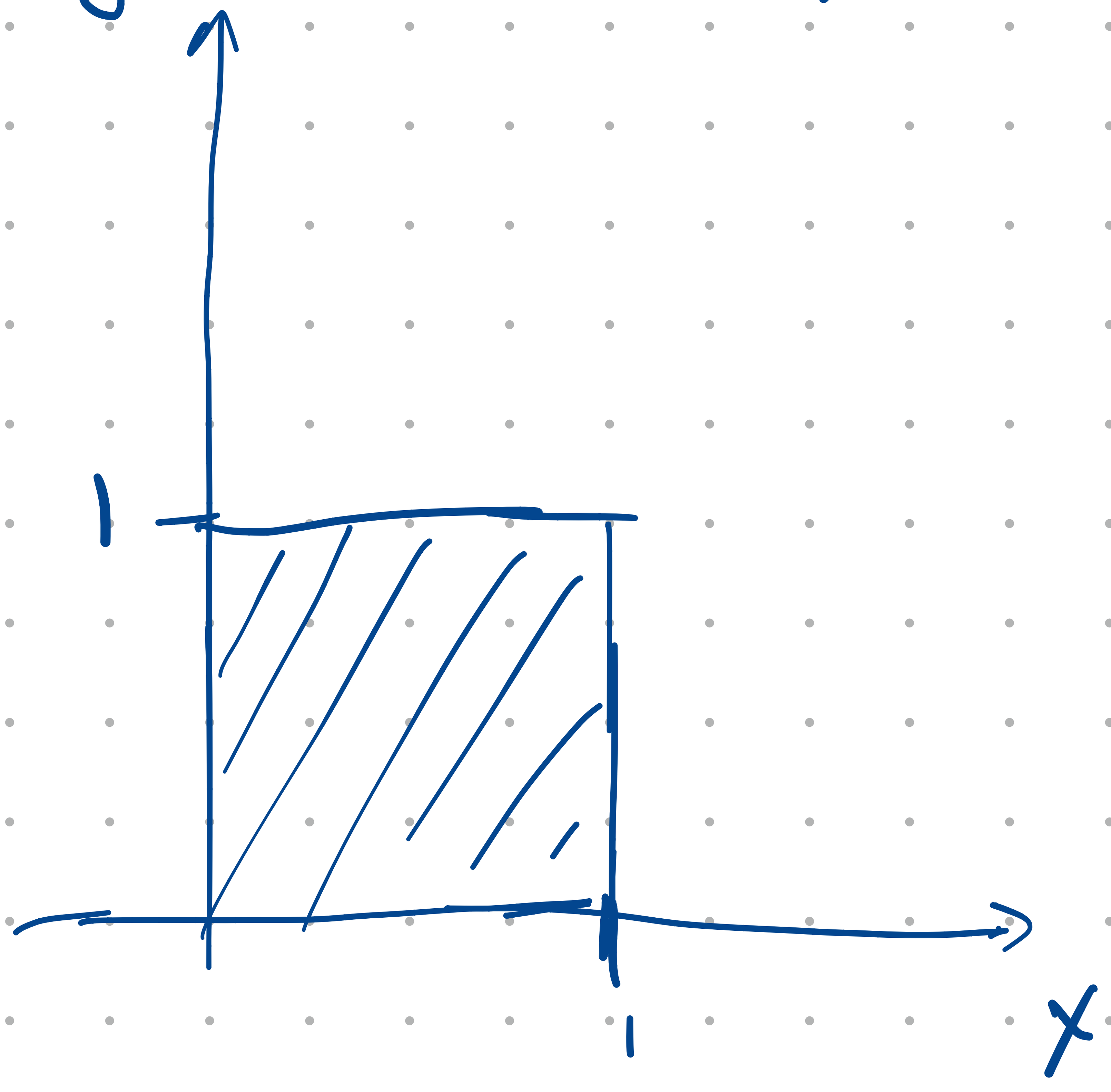
Exercise: rewrite the integral in du dv integration order (may need multiple double integrals)

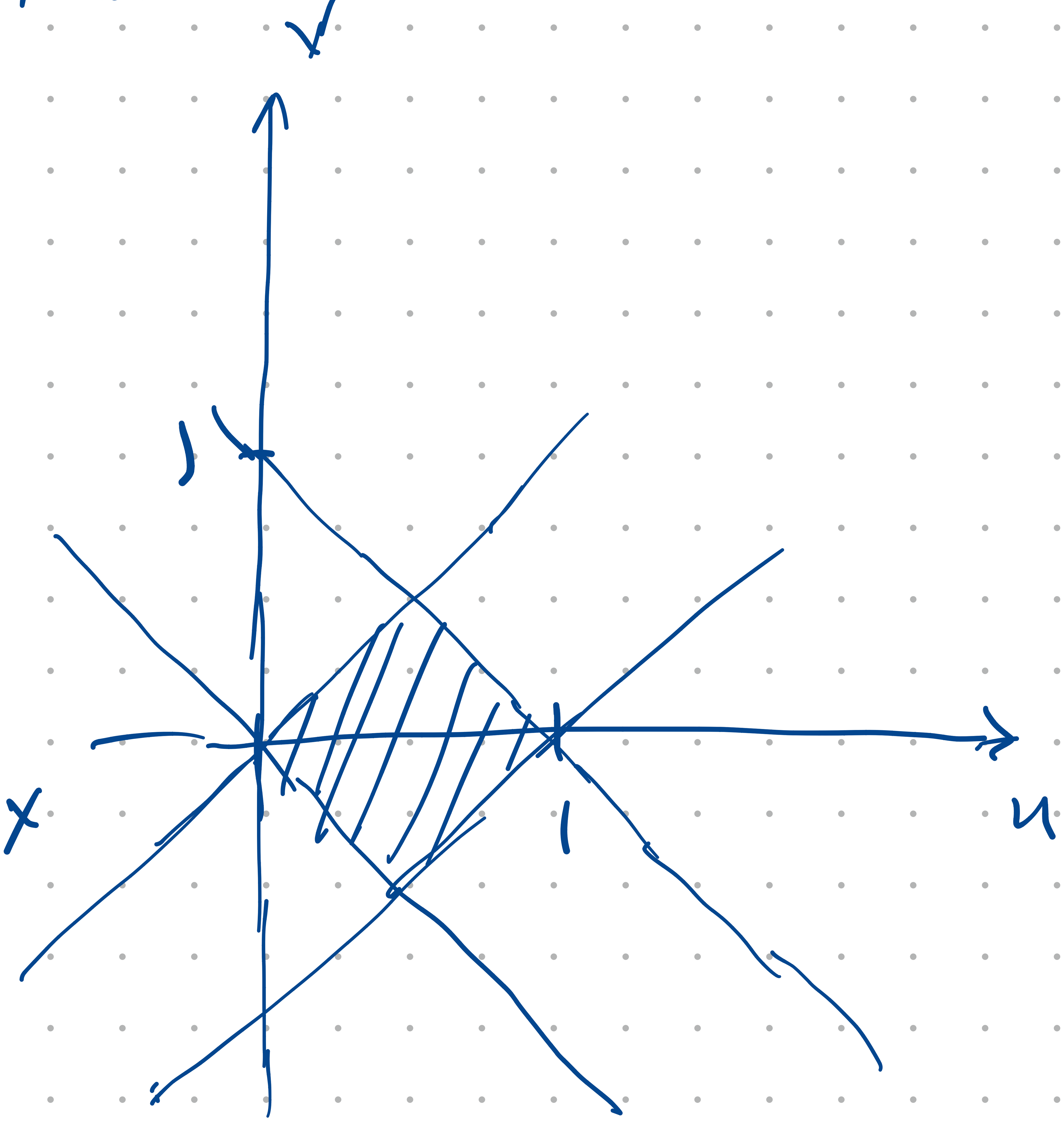
$|\text{Jacobian det}| = 2$

so the integrand is

$$\frac{2}{1-(u+v)(u-v)}\, du\, dv$$

What happens to bounds:





$0 \le x \le 1$

$0 \le y \le 1$

$\rightsquigarrow$

$0 \le u+v \le 1$

$0 \le u-v \le 1$

$$\int_0^{1/2}\int_v^{1-v} \cdots \, du\,dv + \int_{-1/2}^{0}\int_{-v}^{1+v} \cdots \, du\,dv$$

$$= \cdots\cdots = \pi^2/6$$